Numerical Optimization of Large Interconnected Systems

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This paper describes a numerical procedure for optimizing a large set of interconnected systems with respect to a given index of performance. The method automatically constructs the vector differential equation of the complete system from (a) the vector differential equations describing the behavior of the components of the system and (b) the manner of their interconnection. A hill-climbing method is then used to select optimum values of the system's parameters to maximize a given index of performance. An example of the procedure applied to a high order system is included.

The optimization of a system consisting of a large set of interconnected dynamic subsystems is considered in this paper. It is desired to find optimal values of the variable parameters occurring in the system so that a given index of performance is maximized. Such optimization problems occur often in chemical plant design and process control. In these systems the number of equations involved is usually large (for example, a chemical plant often has hundreds of differential equations describing it) and many constraints are imposed on the system variables. It is therefore usually impractical to apply dynamic programming or the maximum principle (1) directly to these systems. Our approach in this paper will be to hill-climb directly on the variable parameters of the system to maximize the given index of performance.

In order to obtain the equations of the complete system explicitly, it is necessary to have a computer algorithm which automatically generates the equations of the system directly from a description of the system. Shannon (2) has developed such a program for steady state chemical engineering design purposes. The basis of his program called PACER is an iteration on steady state values of the system. However, the iteration may not always converge. Sargent (3) is developing similar computer schemes. This paper describes a computer algorithm which automatically constructs the vector differential equation of the complete system from the vector differential equations describing the behavior of the components of the system and the manner of their interconnection. Having obtained the equations of the complete system, standard numerical techniques such as Runge-Kutta integration, may be used to determine the dynamic behavior of the system. If the steady state values of the system are desired, an iterative technique such as is described in Appendix I may be used.

A hill-climbing method is then used to select values of the system's parameters to maximize an index of performance. The index of performance may be any function of the system response which can be numerically computed from the vector differential equation of the complete system. Inequality constraints on functions of the system response and/or variable parameters of the system can be included during this hill-climbing process.

Examples of the use of these methods to optimize a high order system is presented. The example (see Appendix II) involves the steady state optimization of a nonlinear 33rd-order system with seven manipulated variables and eleven constraints.

DEVELOPMENT

The state of a dynamic system is the smallest collection of numbers which must be specified at time $t = t_0$ in order to be able to predict uniquely the behavior of the system for any time $t \geq t_0$. Such numbers are called the *state variables* of the system (4).

A physical system can often be described by the following state equations (4):

$$\dot{\mathbf{x}}(t) = \mathbf{A}[\mathbf{x}, t]\mathbf{x}(t) + \mathbf{B}[\mathbf{x}, t]\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) \qquad (1)$$

where $\mathbf{x}(t)$, an n dimensional vector, is the state of the system at time t, $\mathbf{u}(t)$ is the input to the system, $\mathbf{y}(t)$ is the output of the system and $\mathbf{A}$, $\mathbf{B}$ are matrices which are single-valued functions of the state variables and time. $\mathbf{A}$ is a matrix of order $n \times n$, $\mathbf{C}$ is a matrix of order $m \times n$, $m \leq n$ with rank m and $\mathbf{B}$ is a matrix of order $n \times l$. $\mathbf{u}(t)$ is an l dimensional vector and $\mathbf{y}(t)$ is an m dimensional vector.

It shall therefore be assumed that the state equations of a subsystem of the interconnected system are given by Equation (1).

These equations can always be converted by a change of coordinates into the following form:*

$$\dot{\mathbf{z}}(t) = \overline{\mathbf{A}}[\mathbf{z}, t]\mathbf{z}(t) + \overline{\mathbf{B}}[\mathbf{z}, t]\mathbf{u}(t) \qquad (2)$$
$$\mathbf{y}(t) = (\mathbf{I}_m \quad \mathbf{0})\,\mathbf{z}(t)$$

where $\mathbf{I}_m$ is the identity matrix of order m and

$$\mathbf{z} = \begin{pmatrix} \mathbf{C} \\ \mathbf{0} \quad \mathbf{I}_{n-m} \end{pmatrix} \mathbf{x} \qquad (3)$$

$$\overline{\mathbf{A}} = \begin{pmatrix} \mathbf{C} \\ \mathbf{0} \quad \mathbf{I}_{n-m} \end{pmatrix} \mathbf{A} \left\{ \begin{pmatrix} \mathbf{C} \\ \mathbf{0} \quad \mathbf{I}_{n-m} \end{pmatrix}^{-1} \mathbf{z}, t \right\} \begin{pmatrix} \mathbf{C} \\ \mathbf{0} \quad \mathbf{I}_{n-m} \end{pmatrix}^{-1} \qquad (4)$$

$$\overline{\mathbf{B}} = \begin{pmatrix} \mathbf{C} \\ \mathbf{0} \quad \mathbf{I}_{n-m} \end{pmatrix} \mathbf{B} \left\{ \begin{pmatrix} \mathbf{C} \\ \mathbf{0} \quad \mathbf{I}_{n-m} \end{pmatrix}^{-1} \mathbf{z}, t \right\} \qquad (5)$$

The state Equation (2) will be used to describe a subsystem of the complete system for the following reasons:

1. The system response in terms of state variables can

* It may be necessary to reorder the elements of the state vector $\mathbf{x}$ in Equation (1) so that the matrix $\begin{pmatrix} \mathbf{C} \\ \mathbf{0}\,\mathbf{I}_{n-m} \end{pmatrix}^{-1}$ exists.

be directly associated with the measurable variables of the plant.

2. Indices of performance of the system can often be expressed directly in terms of such state variables.

3. The state equations of a complete system can be simply generated from the state equations of the system's components as will be demonstrated.

The basis of the method to be used may be illustrated by the example given in Figure 1. This example consists of an interconnected set of dynamic subsystems which might occur in a small chemical plant. It is desired to obtain the equations of the complete system from a knowledge of the state equations of each dynamic subsystem and a knowledge of how the subsystems are interconnected. The equations contained in the blocks of the Figure are the state equations of the subsystems of the system. Output variables of the subsystems which can be measured are contained in the subsystem's state variables [since it has been assumed that the equations describing the subsystems are in the form of Equation (2)]. The matrices A_{11}, b_{11}, c_1 etc., may be constants or single valued functions of the state variables and/or of time. The equations of the complete system may immediately be written as follows:

$$\begin{aligned}
\dot{x}_1 &= A_{11}\,x_1 + b_{11}\,x_5 + b_{12}\,x_2 + b_{13}\,x_3 \\
\dot{x}_2 &= A_{22}\,x_2 + b_{21}\,x_1 \\
\dot{x}_3 &= A_{33}\,x_3 + b_{31}\,x_1 + b_{32}\,x_4 + b_{33}\,x_5 + b_{34}\,x_6 \\
\dot{x}_4 &= A_{44}\,x_4 + b_{41}\,x_2
\end{aligned} \tag{6}$$

and these equations may be written in matrix form as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} A_{11} & b_{12} & b_{13} & 0 \\ b_{21} & A_{22} & 0 & 0 \\ b_{31} & 0 & A_{33} & b_{32} \\ 0 & b_{41} & 0 & A_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} b_{11} & 0 \\ 0 & 0 \\ b_{33} & b_{34} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_5 \\ x_6 \end{bmatrix} \tag{7}$$

or more compactly as

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u} \tag{8}$$

The computer algorithm to be described generates such composite matrices **A** and **B** and composite vectors **x** and **u** of Equation (8) directly from:

1. A coded form of the system block diagram
2. The state equations of the subsystems of the system [in the form of Equation (2)].

When the equations of the complete system are in the form of Equation (8), simulation of the dynamic behavior of the complete system can easily be carried out, for example, by solving Equation (8) using Runge-Kutta integration on a digital computer. Alternately, the steady state values of the complete system may be obtained by solving the following set of nonlinear algebraic equations for **x**:

$$\mathbf{x} = \mathbf{A}^{-1}(\mathbf{x})\ \mathbf{B}(\mathbf{x})\mathbf{u} \tag{9}$$

(Appendix I describes a numerical method which may be used to solve such a set of equations). Optimization of the complete system can then be attempted by hill-climbing on variable parameters of the system so that an index of performance [which can be a function of either the steady-state values obtained in solving Equation (9) or the time solution obtained in solving Equation (8)] is maximized subject to constraints in the system.

The procedure for coding a block diagram is as follows:

1. The blocks are numbered consecutively in arbitrary order starting with 1.
2. The state variable vectors are also numbered consecutively in arbitrary order starting at 1. External inputs to the system are also labeled in this way.
3. The block diagram is then encoded in a process matrix such that the ith row of the matrix describes the connections of the ith block, each element of the row being a state variable vector number associated with the block. Output state vectors are differentiated by a minus sign. For example, the process matrix of Figure 1 is given by:

Block no. 1	−1	2	3	5	0
Block no. 2	−2	1	0	0	0
Block no. 3	−3	1	4	5	6
Block no. 4	−4	2	0	0	0

The program then encodes the position of each state component vector ($\mathbf{x}_i$) into the composite state vector [**x** of Equation (8)] using the process matrix and a knowledge of the number of variables in each terminal state variable vector. The composite vector of inputs to the system [**u** of Equation (8)] is also determined this way.

The matrices [**A** and **B** of Equation (8)] are then directly formed from a knowledge of the equation of the system components [which are written in the form of Equation (2)] and the process matrix.

The algorithm may be summarized as follows:

1. Transform the state equations of the subsystems of the system into the form given by Equation (2).
2. Read in the process matrix for the system.
3. Read in the number of state variables associated with each subsystem of the system. For the example considered in Figure 1:

$\mathbf{x}_i$ has l_i state variables, $i = 1, 2, \ldots, 6$

4. Encode the position of each component state vector into the composite state vector. For the example considered in Figure 1:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} x_5 \\ x_6 \end{bmatrix}$$

dimension of vector $\mathbf{x} = \sum_{i=1}^{4} l_i$, dimension of vector $\mathbf{u} = \sum_{i=5}^{6} l_i$.

5. Form the state equations of the complete system. This can be immediately done knowing the process matrix of the system, the position of each component state vector in the composite state vector and a description of the state equations of each component in the system. For the example considered in Figure 1:

$$\dot{\mathbf{x}} = \begin{bmatrix} A_{11} & b_{12} & b_{13} & 0 \\ b_{21} & A_{22} & 0 & 0 \\ b_{31} & 0 & A_{33} & b_{32} \\ 0 & b_{41} & 0 & A_{44} \end{bmatrix} \mathbf{x} + \begin{bmatrix} b_{11} & 0 \\ 0 & 0 \\ b_{33} & b_{34} \\ 0 & 0 \end{bmatrix} \mathbf{u}$$

It is to be noted that such a procedure can normally only be carried out with a digital computer that has a large high speed memory, that is, a system which has a total of 150 state variables requires a computer of the 32K memory class.

NUMERICAL CALCULATION OF THE INDEX OF PERFORMANCE

A hill-climbing method is now used to obtain optimal values of the variable parameters of the system so that the inequality constraints are satisfied and the index of

performance is maximized. The numerical procedure is as follows:

1. Assume a set of variable parameters for the system.
2. Form the overall state equations of the system.
3. Calculate the index of performance from the overall state equations of the system. Determine if the inequality constraints are satisfied.
4. Determine the next best set of variable parameters for the system.
5. Repeat steps 2, 3, 4 until the optimal set of parameter values have been obtained.

Rosenbrock's method of hill-climbing (5) was used in step 4 to determine the next best set of parameters and in all cases considered, it was found to be quite reliable. It should be noted that in step 1 it is extremely important to choose starting values which are reasonably close to the optimum solution; otherwise problems of local optima will occur.

The performance index considered in this paper is one which is a function of the steady state values of the system. It should be noted, however, that any performance index which is a function of the state variables of the system could be considered by this method. However, in systems of large order (which normally will be the case for systems considered in this paper), it is very important to choose a performance index which can be rapidly evaluated numerically; otherwise, because of the nature of the hill-climbing procedure, computation time to maximize the performance index will be excessive, and it will be virtually impossible to optimize the system.

EXAMPLE OF NUMERICAL PROCEDURE: SMALL ABSORBER PLANT

The example considered is a nonlinear system in steady state and represents the absorber section of a contact sulphuric acid plant. It is taken from Johnson, et al. (6) who obtained steady state values of the system using the PACER program (2).

The purpose of the absorber section of a sulphuric acid plant is to produce sulphuric acid and oleum of high concentration from dilute sulphuric acid and sulphur trioxide gas originating from other parts of the plant. Figure 2 is a block diagram of the system considered. Blocks no. 1 and 2 are the absorbers in which a mixture of gases is passed over a stream of 98% acid in order to dissolve sulphur trioxide gas and to increase the concentration of the acid. Blocks no. 3, 4, 6, and 7 are heat exchangers which are used to control the temperatures of the gases or liquid streams passing through them. Block no. 5 is a pump and a mixer.

Streams 1, 2, 3, and 5 are mixtures of oxygen, nitrogen, sulphur dioxide and sulphur trioxide. Streams 4, 6, 7, 8, 10, 11, and 12 consist of sulphuric acid in various concentrations. Stream 9 consists of dilution water and streams 13 to 16 represent temperatures of the streams leaving the heat exchangers. The useful output products

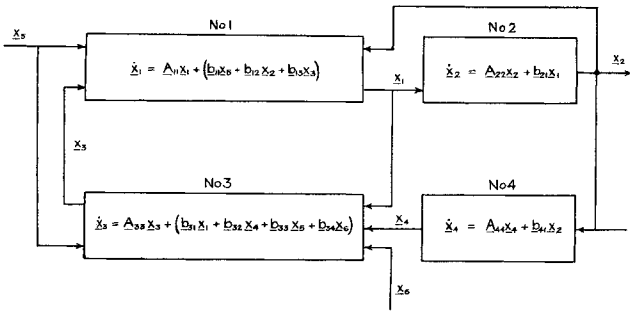

Fig. 1. An interconnected set of dynamic subsystems forming a complete dynamic system.

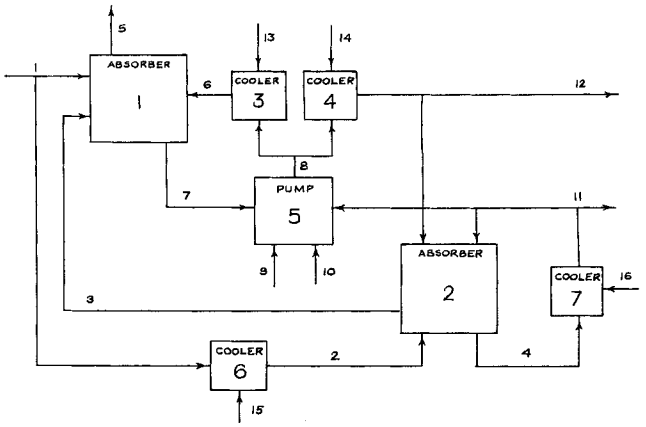

Fig. 2. Block diagram of small absorber plant.

of the absorber section are fed off streams 11 and 12. Stream 5 consists of waste gases.

The process matrix of the plant is given

$$\begin{bmatrix} -5 & -7 & 1 & 3 & 6 & 0 \\ -3 & -4 & 2 & 0 & 11 & 12 \\ -6 & 8 & 13 & 0 & 0 & 0 \\ -12 & 8 & 14 & 0 & 0 & 0 \\ -8 & 7 & 11 & 9 & 10 & 0 \\ -2 & 1 & 15 & 0 & 0 & 0 \\ -11 & 4 & 16 & 0 & 0 & 0 \end{bmatrix}$$

Johnson (6) did not include an index of performance for the plant so that a fictitious index of performance was used in this example. The function used was the value per unit time of the product streams (the portions of streams 11 and 12 bled off) minus the cost per unit time of operating the coolers. (See Appendix II for the exact form of the performance index and for a mathematical description of the plant.)

The following plant and stream parameters were varied in the optimization procedure:

TABLE 1. PARAMETERS AND FLOWS OF PLANT BEFORE AND AFTER OPTIMIZATION

	Before	After
Dilution water entering mixer (block 5)	250 moles/hr.	232 moles/hr.
Outlet stream temperature from cooler 3	80°C.	62°C.
Outlet stream temperature from cooler 6	93°C	95°C.
Relative proportion of stream 8 into cooler 3	0.695	0.777
Relative proportion of stream 8 into cooler 4	0.305	0.223
Relative proportion of stream 12 into absorber (block 2)	0.300	0.859
Relative proportion of stream 11 into bottom output product	0.038	0.613
Relative proportion of stream 11 into absorber (block 2)	0.381	0.006
Relative proportion of stream 11 into mixer (block 5)	0.581	0.381
Top output product concentration (stream 12)	98.6%	98.8%
Top output product flow (stream 12) acid	4,015 moles/hr.	475 moles/hr.
Bottom output product concentration (stream 11)	100%	99.2%
Bottom output product flow (stream 11) oleum	106 moles/hr.	2,740 moles/hr.

1. The relative proportions of stream 8 entering coolers 3 and 4.
2. The proportion of stream 11 recycled into the pump (block 5) and the second absorber (block 2).
3. The proportion of stream 12 recycled into the second absorber (block 2).
4. The outlet stream temperature of coolers 3 and 6.
5. The flow of dilution water (stream 9).

The following variables were constrained:

1. The concentrations of oleum in stream 11 and of acid in stream 12 were constrained to the ranges 99 to 100% and 98 to 100% respectively.
2. Lower limits on the outlet temperatures of coolers 3 and 6 were set at 40 and 50°C. respectively and upper limits at 95°C. in both cases.

The plant parameters found in Johnson (6) were taken as the starting values for the optimization procedure. The index of performance increased from an initial value of 3,682 to a final value of 4,456, an increase of approximately 21% in 6 min. on the IBM 7094 (II) digital computer. Pertinent parameters and flows at the start and at the end of the optimization procedure are given in Table 1. The method of solution used to obtain the steady state values of the system is described in Appendix I.

Table 2 gives typical intermediate print outs of the index of performance and variable parameters during this optimization procedure. It is seen that the hill-climbing process quickly converges to an optimum solution.

TABLE 2. PARAMETERS AND INDEX OF PERFORMANCE DURING OPTIMIZATION PROCEDURE

No. of Trials	Variables parameters (see Appendix II for definition of these parameters) p_1	p_2	p_3	p_4	p_5	p_6	p_7	Index of performance
0	250	80	93	0.581	0.381	0.300	0.695	3,682
56	250	61.8	95	0.268	0.068	0.531	0.926	4,129
75	232	61.6	95	0.381	0.006	0.859	0.777	4,456
96	232	61.6	95	0.381	0.006	0.859	0.777	4,456

CONCLUSIONS

A numerical method has been presented which enables the optimization of a large set of interconnected dynamic components (either linear or nonlinear) to be carried out. The basis of the method is an algorithm* which enables the overall state equations of the system to be obtained directly from the state equations of the components of the system and their manner of interconnection. A hill-climbing technique is then used to maximize a given index of performance of the system with respect to variable parameters of the system.

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* The computer program is available (in Fortran IV) from R. Alas, Design and Development Engineering Department, Westinghouse, Hamilton, Canada.

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APPENDIX I.

Computation of the Steady State of a Nonlinear Dynamic System With a Step-function Input

It is desired to obtain the steady state solutions of the system:

$$\mathbf{x} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u}(t) \tag{1A}$$

for the case that $\mathbf{u}(t)$ is a unit step function input. The steady state solution is given by solving the n nonlinear algebraic equations for $\mathbf{x}$:

$$\mathbf{x} = -\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{u} \tag{2A}$$

The following iterative procedure can be used to solve these equations:

$$\mathbf{x}_{i+1} = [\mathbf{I} - \mathbf{\Gamma}(\mathbf{x}_i)]\mathbf{x}_i + \mathbf{\Gamma}(\mathbf{x}_i)\,[-\mathbf{A}^{-1}(\mathbf{x}_i)\,\mathbf{B}(\mathbf{x}_i)\mathbf{u}] \tag{3A}$$

which will always converge if $\mathbf{\Gamma}(\mathbf{x}_i)$ is chosen to be the inverse of the Jacobian matrix of $\mathbf{x}_i + \mathbf{A}^{-1}(\mathbf{x}_i)\mathbf{B}(\mathbf{x}_i)\mathbf{u}$ and if the starting values $\mathbf{x}_o$ are close enough to the correct solution. It was often found however that if $\mathbf{\Gamma}(\mathbf{x}_i)$ was chosen to be a diagonal matrix

$$\mathbf{\Gamma}(\mathbf{x}_i) = k\mathbf{I} \tag{4A}$$

where $k = 1.0$, satisfactory convergence occurred in the iteration. The 33 simultaneous nonlinear algebraic equations occurring in the example considered were solved in 0.09 min. on the IBM 7094 II digital computer using this method.

APPENDIX II.

Description of Small Absorber Plant

Superscripts on variables correspond to the stream numbers in the block diagram of Figure 2.

Streams consisting of a mixture of gases (no. 1, 2, 3, and 5) have the following variables:

x_1 = stream temperature (°C.)

x_2 = moles/hr. (oxygen)

x_3 = moles/hr. (nitrogen)

x_4 = moles/hr. (sulfur dioxide)

x_5 = moles/hr. (sulfur trioxide)

Streams consisting of liquid (no. 4, 6, 7, 8, 9, 10, 11, and 12) have the following three variables:

x_1 = stream temperature (°C.)

x_2 = moles/hr. (water)

x_3 = moles/hr. (sulfur trioxide in solution)

Streams representing cooler output stream temperature (no. 13, 14, 15, and 16) setting have one variable:

x_1 = output temperature (°C.)

The following parameters are variable:

p_1 = moles/hr. of water in stream no. 9

p_2 = temperature of stream no. 6 (°C.)

p_3 = temperature of stream no. 2 (°C.)

p_4 = proportion of stream no. 11 into pump no. 5

p_5 = proportion of stream no. 11 into absorber no. 2

p_6 = proportion of stream no. 12 into absorber no. 2

p_7 = proportion of stream no. 8 into cooler no. 3

and the following additional variables are constrained:

p_8 = proportion of stream no. 8 into cooler no. 4

p_9 = concentration of stream no. 11

p_{10} = concentration of stream no. 12

p_{11} = compute overflow indicator for linear inversion program used in numerical iteration [see Equation (3A) Appendix I].

Variable p_{11} was introduced and constrained to be ≤ 1 to guarantee that matrix $\mathbf{A}(\mathbf{x}_i)$ [occurring in Equation (3A)

Appendix I] would at all times be nonsingular so that $\mathbf{A}^{-1}(x_i)$ could always be calculated.

Index of Performance

The following index of performance (I.P.) is to be maximized with respect to parameters $p_1, p_2, \ldots, p_7$:

$$\text{I.P.} = 1.2(1-p_4-p_5)x_3^{11} + 0.95(1-p_6)x_3^{12} - R_1 - R_2 - R_3 - R_4$$

where:

$$R_1 = (18\,x_2^{11} + 17\,x_3^{11})[10^{-5}(x_1^4 - x_1^{11}) + 10^{-7}(x_1^4 - x_1^{11})^2]$$

$$R_2 = (18\,x_2^{12} + 17\,x_3^{12})[10^{-5}(x_1^8 - x_1^{12}) + 10^{-7}(x_1^8 - x_1^{12})^2]$$

$$R_3 = (18\,x_2^6 + 17\,x_3^6)[10^{-5}(x_1^8 - x_1^6) + 10^{-7}(x_1^8 - x_1^6)^2]$$

$$R_4 = (7.268\,x_2^2 + 7.036\,x_3^2 + 17.14\,x_4^2 + 17.14\,x_5^2)[10^{-5}(x_1^1 - x_1^2) + 10^{-7}(x_1^1 - x_1^2)^2]$$

R_1, R_2, R_3, and R_4 are empirical functions describing the cost/hr. of operating coolers 7, 4, 3, and 6.

Constraints

The following constraints are imposed:

$$\begin{aligned}
0 &\leq p_1 \leq 1{,}000\\
40 &\leq p_2 \leq 95\\
50 &\leq p_3 \leq 95\\
0 &\leq p_4 \leq 1\\
0 &\leq p_5 \leq 1\\
0 &\leq p_6 \leq 1\\
0 &\leq p_7 \leq 1\\
0 &\leq p_8 \leq 1\\
0.99 &\leq p_9 \leq 1\\
0.98 &\leq p_{10} \leq 1\\
0 &\leq p_{11} \leq 1
\end{aligned}$$

The following equations describe the plant in steady state:

Absorber No. 1

$$\left\{\frac{-t(0)}{\int_0^{10} 1.8[T(N)-t(N)]dN - 273}\right\}x_1^5 + \left\{\frac{0.883\,Q_1}{0.883\,Q_1+Q_3}\right\}x_1^1 + \left\{\frac{Q_3}{0.883\,Q_1+Q_3}\right\}x_1^3 = 0 \quad (1B)$$

$$-x_2^5 + 0.883\,x_2^1 + x_2^3 = 0 \quad (2B)$$

$$-x_3^5 + 0.883\,x_3^1 + x_3^3 = 0 \quad (3B)$$

$$-x_4^5 + 0.883\,x_4^1 + x_4^3 = 0 \quad (4B)$$

$$\left\{\frac{-F(0)}{\int_0^{10} F(N)dN}\right\}x_5^5 + \left\{\frac{0.883}{0.883(x_2^1+x_3^1+x_4^1)+(x_2^3+x_3^3+x_4^3)}\right\}x_5^1 + \left\{\frac{1}{0.883(x_2^1+x_3^1+x_4^1)+(x_2^3+x_3^3+x_4^3)}\right\}x_5^3 = 0 \quad (5B)$$

$$x_1^6 + \left\{\frac{-\int_0^{10}\left[\frac{1.8\gamma}{21.0}[1.176_{10} - 11\,t^2(N) + 1.400_{10} - 5\,t(N) + 7.1][T(N)-t(N)] - 1.63_{10}\,4\,F(N)\right]dN - 273}{T(0)-273}\right\}x_1^1 = 0 \quad (6B)$$

$$-x_2^7 + x_2^6 = 0 \quad (7B)$$

$$-x_5^5 - x_3^7 + 0.883\,x_5^1 + x_5^3 + x_3^6 = 0 \quad (8B)$$

where:

$$Q_1 = (7.268\,x_2^1 + 7.036\,x_3^1 + 17.14\,x_4^1 + 17.14\,x_5^1)$$

$$Q_3 = (7.268\,x_2^3 + 7.036\,x_3^3 + 17.14\,x_4^3 + 17.14\,x_5^3)$$

$$t(0) = \frac{0.883\,Q_1\,x_1^1 + Q_3\,x_1^3}{0.883\,Q_1 + Q_3} + 273$$

$$T(0) = x_1^7 + 273$$

$$F(0) = \frac{0.883\,x_5^1 + x_5^3}{0.883(x_2^1+x_3^1+x_4^1)+(x_2^3+x_3^3+x_4^3)}$$

where

$$\gamma = \frac{0.883(x_2^1+x_3^1+x_4^1)+(x_2^3+x_3^3+x_4^3)}{x_3^6}$$

and N is a spatial variable of the absorber vessel. The integrals occurring in (1B), (5B) and (6B) were evaluated numerically. [The functions $t(N)$, $T(N)$ and $F(N)$ were obtained from (6)].

Absorber No. 2

$$\left\{\frac{-t(0)}{\int_0^{3.75} 1.8[T(N)-t(N)]dN - 273}\right\}x_1^3 + x_1^2 = 0 \quad (9B)$$

$$-x_2^3 + x_2^2 = 0 \quad (10B)$$

$$-x_3^3 + x_3^2 = 0 \quad (11B)$$

$$-x_4^3 + x_4^2 = 0 \quad (12B)$$

$$\left\{\frac{-F(0)}{\int_0^{3.75}[F(N)-F_e]dN}\right\}x_5^3 + \left\{\frac{1}{x_2^2+x_3^2+x_4^2}\right\}x_5^2 = 0 \quad (13B)$$

$$\left\{\frac{\int_0^{3.75}\left[\frac{1.8\gamma}{32.5}[1.176_{10} - 11\,t^2(N) + 1.4_{10} - 5\,t(N) + 7.1][T(N)-t(N)] - 1.6_{10}\,4[F_e - F(N)]\right]dN - 273}{T(0)-273}\right\}x_1^4 + \left[\frac{p_5 Q_{11}}{p_5 Q_{11} + p_6 Q_{12}}\right]x_1^{11} + \left[\frac{p_6 Q_{12}}{p_5 Q_{11} + p_6 Q_{12}}\right]x_1^{12} = 0 \quad (14B)$$

$$-x_2^4 + p_5\,x_2^{11} + p_6\,x_2^{12} = 0 \quad (15B)$$

$$-x_5^3 - x_3^4 + p_5\,x_3^{11} + p_6\,x_3^{12} = 0 \quad (16B)$$

where:

$$Q_{11} = (18\,x_2^{11} + 17\,x_3^{11})$$

$$Q_{12} = (18\,x_2^{12} + 17\,x_3^{12})$$

$$t(0) = x_1^2 + 273$$

$$T(0) = x_1^7 + 273$$

$$F(0) = \frac{p_5 x_5^{11} + p_6 x_5^{12}}{p_5(x_2^{11} + x_3^{11} + x_4^{11}) + p_6(x_2^{12} + x_3^{12} + x_4^{12})}$$

$$F_e = \frac{1}{770} \cdot {}_{10}[-0.26 + 0.03\,(T(N) - 273)]$$

where:

$$\gamma = \frac{2\,p_5(x_2^{11} + x_3^{11} + x_4^{11}) + 2\,p_6(x_2^{12} + x_3^{12} + x_4^{12})}{p_5(x_2^{11} + x_3^{11}) + p_6(x_2^{12} + x_3^{12})}$$

and N is a spatial variable of the absorber vessel. The integrals occurring in (9B), (13B), and (14B) were evaluated numerically.

Cooler No. 3

$$x_1^6 = x_1^{13} = P_2 \tag{17B}$$

$$x_2^6 = p_7 \cdot x_2^8 \tag{18B}$$

$$x_3^6 = p_7 \cdot x_3^8 \tag{19B}$$

Cooler No. 4

$$x_1^{12} = x_1^{14} \tag{20B}$$

$$x_2^{12} = (1 - p_7)x_2^8 \tag{21B}$$

$$x_3^{12} = (1 - p_7)x_3^8 \tag{22B}$$

Cooler No. 6

$$x_1^2 = x_1^{15} = p_3 \tag{23B}$$

$$x_2^2 = 0.117\,x_2^1 \tag{24B}$$

$$x_3^2 = 0.117\,x_3^1 \tag{25B}$$

$$x_4^2 = 0.117\,x_4^1 \tag{26B}$$

$$x_5^2 = 0.117\,x_5^1 \tag{27B}$$

Cooler No. 7

$$x_1^{11} = x_1^{16} \tag{28B}$$

$$x_2^{11} = x_2^4 \tag{29B}$$

$$x_3^{11} = x_3^4 \tag{30B}$$

Pump No. 5

$$-x_1^8 + \left[\frac{Q_7}{Q + p_2 Q_{11} + Q_9 + Q_{10}}\right]x_1^7 + \left\{\frac{P_2 Q_{11}}{Q_7 + p_2 Q_{11} + Q_9 + Q_{10}}\right\}x_1^{11} + \left\{\frac{Q_{10}}{Q^7 + p_2 Q_{11} + Q_9 + Q_{10}}\right\}x_1^{10} = 0 \tag{31B}$$

$$\left\{\frac{Q_9}{Q^7 + p_2 Q_{11} + Q_9 + Q_{10}}\right\}x_1^5$$

$$-x_2^8 + x_2^7 + p_2 \cdot x_2^{11} + x_1^9 + x_2^{10} = 0 \tag{32B}$$

$$-x_3^8 + x_3^7 + p_2 \cdot x_3^{11} + x_3^{10} = 0 \tag{33B}$$

where:

$$Q_7 = 18\,x_2^7 + 17\,x_3^7$$

$$Q_{11} = p_2(18\,x_2^{11} + 17\,x_3^{11})$$

$$Q_9 = 18\,x_1^9$$

$$Q_{10} = 18\,x_2^{10} + 17\,x_3^{10}$$

Effects of Ultrasonic Vibrations on Heat Transfer to Liquids by Natural Convection and by Boiling

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The effects of ultrasonic vibrations on heat transfer to water and methanol by natural convection and by boiling were measured at three ultrasonic energy levels with frequency ranging from 20.6 to 306 kcycles/sec., using electrically heated platinum wires of diameters 0.007 and 0.010 in. Up to an eight-fold increase in heat transfer coefficient was obtained in natural convection, but the effects diminished with increased temperature difference and became negligible in the well-developed nucleate boiling region. High-speed photographs showed that the increase was due to the motion of cavitation bubbles on the wire surface. The heat transfer results were correlated by local cavitation activity values measured by a technique developed for this work.

Acoustic cavitation usually refers to the formation of small bubbles and their subsequent growth and collapse within a liquid sound field. Such cavitation bubbles can be used to create disturbances in the relatively stagnant film of liquid near a heating surface as well as to increase the bulk mixing in the liquid. Thus up to a four-fold increase in natural convective heat transfer coefficient was reported by Larson and London (*1*). Similar study by Robinson, et al. (*2*), however, showed that the increase was less than 20%. Direct comparison of published data is usually impossible because of the various methods used to measure and report the acoustic properties of the system. The effects of ultrasonic vibrations in many cases may also depend on the location of the heating surface in the ultrasonic field. The results will be strongly dependent on the geometry of the test system.

The limited number of papers published on the effects of ultrasonic vibrations on boiling heat transfer is mainly concerned with the increase of burnout heat flux. Isakoff (*3*) appears to be the only investigator to have studied the effect on nucleate pool boiling and reported that the range of nucleate boiling could be considerably extended. Romie and Aronson (*4*) as well as Ornatskii and Shcherbakov (*5*) suggested that the effect of ultrasonic vibrations would be to reduce the mean bubble size and to increase the frequency of bubble formation. No experi-

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